

Instanton test of non-supersymmetric deformations of the $AdS_5 \times S^5$

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ABSTRACT: We consider instanton effects in a non-supersymmetric gauge theory obtained by marginal deformations of the $\mathcal{N} = 4$ SYM. This gauge theory is expected to be dual to type IIB string theory on the AdS_5 times deformed- S^5 background. From an instanton calculation in the deformed gauge theory we extract the prediction for the dilaton-axion field τ in dual string theory. In the limit of small deformations where the supergravity regime is valid, our instanton result reproduces the expression for τ of the supergravity solution found by Frolov.

KEYWORDS: Supersymmetric gauge theory, AdS-CFT Correspondence, Solitons Monopoles and Instantons.

Contents

| | |
|--|-----------|
| 1. Introduction and discussion of results | 1 |
| 2. Three-parameter deformation of the $AdS_5 \times S^5$ | 2 |
| 2.1 Supergravity dual | 3 |
| 2.2 Gauge theory formulation | 3 |
| 3. Instanton effects | 6 |
| 4. Complex β deformations | 10 |

1. Introduction and discussion of results

Much has been learnt recently about the gauge theory – string theory duality by investigating how the AdS/CFT correspondence [1] is realised when the $\mathcal{N} = 4$ supersymmetric gauge theory is deformed by exactly marginal operators [2–17]. Since the gauge theory stays conformal it is expected to be dual (in the appropriate limit) to a supergravity solution with Anti-de Sitter geometry. There are two points which make these marginal deformations particularly interesting. First, is that these deformations give a continuous family of theories parameterised by the deformation parameters β_i . The AdS/CFT duality provides a mapping between a gauge theory and a string theory for each value of β_i . By studying the β -dependence in gauge theory and in the dual supergravity (or string theory) one thus gets a more detailed understanding of the AdS/CFT correspondence. The second feature of marginal β -deformations is that they break (partially or completely) the supersymmetry of the original $\mathcal{N} = 4$ theory.

Lunin and Maldacena [2] have found a supergravity dual of the β -deformed $\mathcal{N} = 4$ super Yang-Mills theory (β -SYM) which preserves $\mathcal{N} = 1$ supersymmetry. In a recent paper [15] two of the present authors have tested this supergravity solution and the resulting string theory effective action against an instanton calculation on the gauge theory side. It was found in [15] that the correct expression for the dilaton-axion supergravity field τ was reproduced by instanton effects in gauge theory, and that the higher-derivative terms in the string theory effective action included the appropriate modular forms $f_n(\tau, \bar{\tau})$ of this τ as one would expect from the $SL(2, Z)$ duality in IIB string theory.

One way of realising the solution generating method in [2] is via a combined T-duality-shift-T-duality (TsT) transformation of the supergravity $AdS_5 \times S^5$ geometry. This approach enabled Frolov [7] to extend the method and to find a three-parameter family of non-supersymmetric supergravity solutions. This background has to be AdS/CFT dual to a non-supersymmetric conformal gauge theory obtained by a certain three-parameter deformation of the $\mathcal{N} = 4$ SYM.

In this paper we apply the instanton approach of refs. [18, 15] to investigate this non-supersymmetric gauge theory and to test the supergravity solution of Ref. [7]. In section **2** we write down the supergravity solution of [7] parameterised by three real deformations γ_i and specify the corresponding γ_i -deformed gauge theory. In section **3** we carry out an instanton calculation in the γ_i -deformed gauge theory with a view to reconstruct the dilaton-axion supergravity field τ from gauge theory. In the appropriate double-scaling limit, $\gamma_i \ll 1$, our result

$$\tau = \tau_0 + 2N\pi i (\gamma_3^2 \mu_1^2 \mu_2^2 + \gamma_1^2 \mu_2^2 \mu_3^2 + \gamma_2^2 \mu_3^2 \mu_1^2) \tag{1.1}$$

reproduces the τ -field of Frolov’s supergravity dual. Here τ_0 is the usual complexified coupling constant in gauge theory, γ_i are the three deformation parameters, and μ_i are coordinates on the deformed S^5 sphere in supergravity. In section **4** we generalise our set-up to include complex-valued deformations $\beta_i = \gamma_i + i\sigma_i$. Our main results there are eqs. (4.6)-(4.9).

The fact that instanton contributions in gauge theory confirm the non-supersymmetric supergravity solution of Ref. [7] is our main result. Both expressions, in gauge theory and in supergravity, are continuous functions of the three complex deformation parameters. What is interesting about this matching is not merely the fact that there is a non-trivial agreement between gauge theory and supergravity, but also that the Yang-Mills instanton calculation which is intrinsically valid only at weak coupling, $g^2N \ll 1$, $N \rightarrow \infty$, appears to give the correct result in the strong coupling limit, $g^2N \gg 1$, relevant for comparison with the supergravity. This agreement between the strong and the weak coupling limits is completely analogous to the previously known instanton tests of AdS/CFT correspondence in the $\mathcal{N} = 4$ SYM context in refs. [19–22, 18, 23, 24] and more recently in the context of supersymmetry-preserving β -deformations in ref. [15]. In all known cases, leading order contributions of Yang-Mills instantons calculated at $g^2N \ll 1$, match with contributions of D-instantons in supergravity in the opposite limit $g^2N \gg 1$. The agreement holds only for the instanton part of the answer, it is known that perturbative contributions in gauge theory and in string theory do not match. This suggests that there should exist a non-renormalisation theorem which would apply to the instanton effects and explain the agreement. We refer the reader to refs. [18, 25] and [15, 24] for a more detailed discussion on this point.

In this paper we find that the agreement in the instanton sector persists in the non-supersymmetric case. This implies that the non-renormalisation theorem is not dictated by supersymmetry. We expect that the origin of the agreement lies in identifying Yang-Mills instantons with D-instantons as the ‘extended’ objects or defects in both theories.

2. Three-parameter deformation of the $AdS_5 \times S^5$

We begin by reviewing the theories on each side of the gauge/string duality we wish to study. The solution generating tool on the supergravity side is the combination of T-dualities and coordinate shifts known as a TsT transformation. These allow one to start with the known duality between IIB supergravity on a flat background and $\mathcal{N} = 4$ SYM,

and generate new supergravity backgrounds [2, 7]. The deformation on the gauge theory side will be incorporated by introducing an appropriate star-product between fundamental fields. For the most part we will concern ourselves with real valued deformations of the theory. The issues which arise for complex deformations will be discussed in section 4.

2.1 Supergravity dual

In order to perform supergravity TsT transformations one must first identify suitable tori in the initial geometry. In the case of [2] this torus was chosen to be the one dual to the $U(1) \times U(1)$ global symmetry of β -SYM. If we parameterise this torus with angular variables (φ_1, φ_2) , then a TsT transformation with parameter $\hat{\gamma}$ is the following: T-dualise in the φ_1 direction, perform the shift $\varphi_2 \rightarrow \varphi_2 + \hat{\gamma}\varphi_1$, T-dualise again along φ_1 . The resulting supergravity solution was shown in [2] to be dual to β -SYM for small, real β under the association $\hat{\gamma} = R^2\beta$ where R is the radius of S^5 .

The S^5 factor of $AdS_5 \times S^5$ can be parameterised with the coordinates μ_1, μ_2, μ_3 with $0 \leq \mu_i \leq 1$ subject to $\mu_1^2 + \mu_2^2 + \mu_3^2 = 1$ and the angular coordinates ϕ_1, ϕ_2, ϕ_3 . There are clearly three independent choices of torus corresponding to the pairs $(\phi_1, \phi_2), (\phi_2, \phi_3)$ and (ϕ_1, ϕ_3) . The three parameter deformation constructed in ref. [7] follows by performing a separate TsT transformation on each of these, with shift parameters $\hat{\gamma}_3, \hat{\gamma}_1$ and $\hat{\gamma}_2$ respectively. The resulting type IIB supergravity background of Frolov written in string frame with $\alpha' = 1$ takes the form [7]:

$$\begin{aligned}
 ds_{\text{str}}^2 &= R^2 \left[ds_{\text{AdS}}^2 + \sum_i (d\mu_i^2 + G \mu_i^2 d\phi_i^2) + G \mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_i \hat{\gamma}_i d\phi_i \right)^2 \right], \quad (2.1) \\
 G^{-1} &= 1 + \hat{\gamma}_3^2 \mu_1^2 \mu_2^2 + \hat{\gamma}_1^2 \mu_2^2 \mu_3^2 + \hat{\gamma}_2^2 \mu_3^2 \mu_1^2, \quad e^{2\phi} = e^{2\phi_0} G, \\
 B^{NS} &= R^2 G (\hat{\gamma}_3 \mu_1^2 \mu_2^2 d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1 \mu_2^2 \mu_3^2 d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2 \mu_3^2 \mu_1^2 d\phi_3 \wedge d\phi_1)
 \end{aligned}$$

We present here only the fields that will be relevant for our purposes. The full complement, including the RR forms C_2 and C_4 and self-dual five-form fields is given in [7]. To make contact with the dual gauge theory we have the usual AdS/CFT relation $R^4 \equiv 4\pi e^{\phi_0} N = \sqrt{\lambda}$. The real deformation parameters $\hat{\gamma}_i$ appearing in (2.1) are related to the γ_i deformations on the gauge theory side via a simple rescaling, $\hat{\gamma}_i = R^2\gamma_i$. We note that the dilaton field ϕ in (2.1) is not simply a constant, but depends on the coordinates of the deformed sphere \tilde{S}^5 . (The axion field $C = C^0$ is a constant for real-valued deformations γ_i .)

When all three deformation parameters are equal, $\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma}$, this solution reverts to that of Lunin and Maldacena [2], and the dual gauge theory is β -SYM.

2.2 Gauge theory formulation

The Frolov’s supergravity solution (2.1) with three real deformations γ_i contains the AdS_5 factor. Thus it is expected to be dual to a conformal gauge theory obtained by exactly marginal but non-supersymmetric deformations of the $\mathcal{N} = 4$ SYM. More precisely, the gauge theory should be conformal in the large number of colours limit (which we always assume in this paper) where the supergravity approximation to string theory can be trusted.

| | Φ_1 | Φ_2 | Φ_3 | A_μ | λ_1 | λ_2 | λ_3 | λ_4 |
|-------|----------|----------|----------|---------|----------------|----------------|----------------|----------------|
| Q_1 | 1 | 0 | 0 | 0 | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| Q_2 | 0 | 1 | 0 | 0 | $-\frac{1}{2}$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| Q_3 | 0 | 0 | 1 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ | $+\frac{1}{2}$ |

Table 1: Charges Q_i of the component fields in the theory under the Cartan subgroup of the $SU(4)_R$.

We will be considering non-supersymmetric deformations of the $\mathcal{N} = 4$ gauge theory, parameterised by three phases, $e^{i\pi\gamma_1}$, $e^{i\pi\gamma_2}$ and $e^{i\pi\gamma_3}$, with real parameters γ_i . To ensure conformal invariance of the theory in the large N limit, it is convenient to introduce these phase-deformations via a star-product approach.¹ We take the component Lagrangian of the $\mathcal{N} = 4$ supersymmetric Yang-Mills and modify all products of fields there into star-products. For any pair of fields f and g , the star-product which gives rise to our deformations is [10]:

$$f * g \equiv e^{-i\pi Q_i^f Q_j^g \epsilon_{ijk} \gamma_k} fg \tag{2.2}$$

Here Q_i^f and Q_i^g are the charges of the fields f and g under the $i = 1, 2, 3$ Cartan generators of the $SU(4)_R$ R-symmetry of the original $\mathcal{N} = 4$ SYM. The values of these charges for all component fields are the same as in [9] and are given in the table 1. These values are easy to derive from the fact that the integral of the superpotential of the $\mathcal{N} = 4$ SYM

$$\int d^2\theta \mathcal{W}_{\mathcal{N}=4} = \int d^2\theta i \text{Tr}(\Phi_1 \Phi_2 \Phi_3 - \Phi_1 \Phi_3 \Phi_2) \tag{2.3}$$

is invariant under the action of each of these Cartan generators on the superfields Φ_i

$$\Phi_1 \rightarrow e^{i\phi_1} \Phi_1, \quad \Phi_2 \rightarrow e^{i\phi_2} \Phi_2, \quad \Phi_3 \rightarrow e^{i\phi_3} \Phi_3 \tag{2.4}$$

This implies that the Grassmann $\mathcal{N} = 1$ superspace coordinate θ_α is charged under these transformations with $Q^\theta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The charges of the scalar fields Φ_i are precisely the same as of their parent superfields Φ_i in (2.4) and the charges of the fermions λ_A in the table 1 are read from eq. (2.4) keeping in mind $\Phi_i(x, \theta) = \Phi_i(x) + \theta \cdot \lambda_i(x) + \dots$. The gauge field A_μ is neutral.²

The Lagrangian of the deformed theory follows from the component Lagrangian of the $\mathcal{N} = 4$ SYM and the star-product definition (2.2). We have:

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D^\mu \bar{\Phi}^i)(D_\mu \Phi_i) - \frac{1}{2} [\Phi_i, \Phi_j]_{C_{ij}} [\bar{\Phi}^i, \bar{\Phi}^j]_{C_{ij}} + \frac{1}{4} [\Phi_i, \bar{\Phi}^i][\Phi_j, \bar{\Phi}^j] \right)$$

¹Below we will use the fact [26] that associative star-products do not change planar diagrams of the original $\mathcal{N} = 4$ SYM. This means that in the large N limit the resulting deformed gauge theory will remain conformal at least in perturbation theory.

²The fourth fermion λ_4 is the $\mathcal{N} = 1$ superpartner of A_μ . It's charge is read off the invariance of the gauge kinetic term, $\int d^2\theta WW$, where $W_\alpha = \lambda_{4\alpha} + \dots$ is the usual field-strength chiral superfield.

$$+ \lambda_A \sigma^\mu D_\mu \bar{\lambda}^A - i[\lambda_4, \lambda_i]_{B_{4i}} \bar{\Phi}^i + i[\bar{\lambda}^4, \bar{\lambda}^i]_{B_{4i}} \Phi_i + \frac{i}{2} \epsilon^{ijk} [\lambda_i, \lambda_j]_{B_{ij}} \Phi_k + \frac{i}{2} \epsilon_{ijk} [\bar{\lambda}^i, \bar{\lambda}^j]_{B_{ij}} \bar{\Phi}^k \Big) \tag{2.5}$$

This Lagrangian contains only ordinary products between the fields; all modifications due to the star-product (2.2) are assembled in (2.5) into the deformed commutators of scalars Φ_i , $\bar{\Phi}^i$ and fermions λ_A , $\bar{\lambda}^A$. These deformed commutators are

$$[\Phi_i, \Phi_j]_{C_{ij}} := e^{iC_{ij}} \Phi_i \Phi_j - e^{-iC_{ij}} \Phi_j \Phi_i, \quad i, j = 1, 2, 3 \tag{2.6}$$

$$[\lambda_A, \lambda_B]_{B_{AB}} := e^{iB_{AB}} \lambda_A \lambda_B - e^{-iB_{AB}} \lambda_B \lambda_A, \quad A, B = 1, \dots, 4 \tag{2.7}$$

Deformed commutators for $\bar{\Phi}$ and $\bar{\lambda}$ fields defined in the same way as in (2.6)-(2.7), and we note that the commutator $[\Phi_i, \bar{\Phi}^i]$ in (2.5) is undeformed. The matrices C and B are the same as in ref. [9], they read

$$C = \pi \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}, \quad B = \pi \begin{pmatrix} 0 & -\frac{1}{2}(\gamma_1 + \gamma_2) & \frac{1}{2}(\gamma_3 + \gamma_1) & \frac{1}{2}(\gamma_2 - \gamma_3) \\ \frac{1}{2}(\gamma_1 + \gamma_2) & 0 & -\frac{1}{2}(\gamma_2 + \gamma_3) & \frac{1}{2}(\gamma_3 - \gamma_1) \\ -\frac{1}{2}(\gamma_3 + \gamma_1) & \frac{1}{2}(\gamma_2 + \gamma_3) & 0 & \frac{1}{2}(\gamma_1 - \gamma_2) \\ -\frac{1}{2}(\gamma_2 - \gamma_3) & -\frac{1}{2}(\gamma_3 - \gamma_1) & -\frac{1}{2}(\gamma_1 - \gamma_2) & 0 \end{pmatrix} \tag{2.8}$$

We see that the whole effect of the 3-parameter deformation is contained in these matrices which introduce the appropriate phases into the 4-scalar and the Yukawa interactions of the deformed theory (2.5). It is important to note that the induced phases of the fermions (determined by the matrix B) are different from those of the scalars (in C). Also the ranks of B and C are different, the matrix B introduces phases to the Yukawa interactions involving all the fermions, including the gaugino λ_4 . The Lagrangian (2.5) incorporates correctly the four-scalar interactions written down in [7, 10]. In addition to these, eqs. (2.5), (2.8) give the precise form of the interactions with fermions which are required for the instanton calculations in the present paper.

For a special case of all γ_i being equal, the matrices B and C coincide with each other giving the same phase factors for scalars and fermions. In this case, the gauge theory is $\mathcal{N} = 1$ supersymmetric and is dual to the supergravity solution of Lunin and Maldacena [2]. In the general case of unequal deformations γ_i , the fermion and scalar phases differ and the gauge theory is non-supersymmetric.

Finally, we need to make sure that the deformed gauge theory defined by eqs. (2.5), (2.8) is exactly marginal in the large N limit. In general, this would be a non-trivial task since the theory is not supersymmetric and one cannot use the approach of Leigh and Strassler [3] to establish the required conformal invariance. Instead the marginality of the theory follows from the use of the star-product. It is known [26] that the Moyal star-products used in the formulation of the noncommutative field theory do not affect the large N perturbation theory. More precisely, the planar diagrams of the theories with and without the star-products can differ only by an overall phase-factor which depends only on the external lines. This argument essentially uses only the associativity property of the star-product and it also applies to our choice (2.2), see section 3.2 of ref. [27] for more

detail. This implies that all planar perturbative contributions to the beta-functions and anomalous dimensions in our deformed theory are proportional to those in the conformal $\mathcal{N} = 4$ theory, and hence vanish. Thus, the deformed theory (2.5), (2.8) is conformal in the large N perturbation theory.

The deformed theory (2.5), (2.8) is an interesting field theory on its own right. It is a non-supersymmetric theory which fully inherits the remarkable structure of the large N perturbation theory of the superconformal $\mathcal{N} = 4$ SYM. In ref. [28] it was argued that in the $\mathcal{N} = 4$ SYM the Maximally-Helicity-Violating (MHV) n -point amplitudes have an iterative structure, such that the kinematic dependence of all higher-loop MHV amplitudes can be determined from the known one-loop results. It then follows [27] that the same must be true for the planar MHV amplitudes of the deformed theory. This is a consequence of the fact that the deformations were introduced via the star-product of the type (2.2). It is remarkable that such an iterative structure of the multi-loop amplitudes can hold in a non-supersymmetric theory.

3. Instanton effects

Instantons in the deformed $\mathcal{N} = 4$ gauge theory have been discussed in detail in ref. [15]. We refer the reader to this reference and summarise here only the key points. The instanton configuration is defined to the leading order in the Yang-Mills coupling g , and satisfies the following equations for the gauge field,

$$F_{mn} = {}^*F_{mn} \tag{3.1}$$

fermions,

$$\bar{\mathcal{D}}^{\dot{\alpha}\alpha} \lambda_{\alpha}^A = 0 \tag{3.2}$$

and scalars,

$$\mathcal{D}^2 \Phi^{AB} = \sqrt{2} i (e^{iB_{AB}} \lambda^A \lambda^B - e^{-iB_{AB}} \lambda^B \lambda^A) \tag{3.3}$$

Here $\bar{\mathcal{D}}^{\dot{\alpha}\alpha} = D^{\mu} \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}$ and $\mathcal{D}^2 = D^{\mu} D_{\mu}$ where D_{μ} is the covariant derivative in the instanton background A_{μ} . The matrix B is given in (2.8).

There are $8kN$ fermionic solutions of (3.2) in the k -instanton background. 16 of these solutions correspond to $2\mathcal{N} = 8$ supersymmetric and $2\mathcal{N} = 8$ superconformal fermion zero modes of the original $\mathcal{N} = 4$ gauge theory. In the $\mathcal{N} = 4$ SYM these 16 fermion zero modes are exact. In our deformed theory supersymmetry is lost and all of the fermion zero modes are lifted in the instanton action as will be seen shortly.

The scalar field equation (3.3) follows from the Yukawa interactions³ in (2.5) and is written in the basis $\Phi^{AB} = -\Phi^{BA}$ for the scalar fields. This representation is related as follows to the usual basis Φ_i used in (2.5) (see [15])

$$\Phi_1 = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) = 2\bar{\Phi}^{23} = 2\Phi^{14}$$

³The four-scalar interactions in (2.5) do not enter the leading order in g instanton construction.

$$\begin{aligned}\Phi_2 &= \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4) = 2\bar{\Phi}^{31} = 2\Phi^{24} \\ \Phi_3 &= \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6) = 2\bar{\Phi}^{12} = 2\Phi^{34}\end{aligned}\tag{3.4}$$

The instanton configuration is the solution of the defining equations (3.1)-(3.3), and it is used to construct the semi-classical instanton integration measure. This measure is an integral over the instanton collective coordinates and it determines instanton contributions to the path integral. The general multi-instanton measure was constructed in [18] for the $\mathcal{N} = 4$ SYM and generalised in [15] to account for the supersymmetry preserving β -deformations. The result of [15] can be now straightforwardly adopted to the case of non-supersymmetric γ_i deformations. We will concentrate here on the simplest case of the single-instanton measure. The multi-instanton measure is a straightforward generalisation of thereof along the lines of [15, 18].

The 1-instanton measure of the γ_i -deformed theory reads (cf. eq. (5.2) of ref. [15]):

$$\int d\mu e^{-S_{1-\text{inst}}} = \frac{2^{-31}\pi^{-4N-5}g^{4N}}{(N-1)!(N-2)!} \int d^4x_0 d\rho d^6\chi \prod_{A=1}^4 d^2\xi^A d^2\bar{\eta}^A d^{(N-2)}\nu^A d^{(N-2)}\bar{\nu}^A \rho^{4N-7} \exp\left[-\frac{8\pi^2}{g^2} + i\theta - 2\rho^2\chi^a\chi^a + \frac{4\pi i}{g}\chi_{AB}\Lambda_{AB}\right]\tag{3.5}$$

The integral above is over the bosonic and fermionic (Grassmann) collective coordinates of the instanton. The fermionic ones comprise $4(N-2)$ parameters ν_i^A (where $i = 1, \dots, N-2$), 8 supersymmetric coordinates ξ_α^A and 8 superconformal modes $\bar{\eta}_{\dot{\alpha}}^A$ (where $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$). Bosonic collective coordinates include the instanton position x_0^μ , the scale-size ρ and the 6 additional variables χ^a which are coupled to fermion modes in the instanton action in the exponent in (3.5). The variables χ^a or χ_{AB} transform in the vector representation of the $\text{SO}(6) \cong \text{SU}(4)$ R-symmetry and is subject to the reality condition $\chi_{AB}^\dagger = \frac{1}{2}\epsilon_{ABCD}\chi_{CD}$.

Finally Λ_{AB} in the instanton action in (3.5) is a fermionic bilinear defined as

$$\Lambda_{AB} = \frac{1}{2\sqrt{2}} \sum_{i=1}^{N-2} (e^{iB_{AB}} \bar{\nu}_i^A \nu_i^B - e^{-iB_{AB}} \bar{\nu}_i^B \nu_i^A) + i8\sqrt{2} \sin(B_{AB}) (\rho^2 \bar{\eta}^A \cdot \bar{\eta}^B + \xi^A \cdot \xi^B)\tag{3.6}$$

The 4×4 antisymmetric matrix B_{AB} was defined in (2.8). The fact that the instanton action in (3.5) depends on all of the fermionic collective coordinates (through Λ_{AB}) implies that they are lifted. This is to be expected in the non-supersymmetric theory.

Following the approach of [15] we proceed by integrating out fermionic collective coordinates ν_i^A and $\bar{\nu}_i^A$ from the instanton partition function (3.5). For each value of $i = 1, \dots, N-2$ this integration gives a factor of

$$\left(\frac{4\pi}{g} \frac{1}{\sqrt{2}}\right)^4 \det_4(e^{iB_{AB}} \chi_{AB})\tag{3.7}$$

The determinant above can be calculated directly. It will be useful to express the result in terms of the three complex variables X_i which are defined in terms of χ_{AB} in the way

analogous to eqs. (3.4):

$$\begin{aligned}
 X_1 &= \chi^1 + i\chi^2 = 2\sqrt{2}\chi_{23}^\dagger = 2\sqrt{2}\chi_{14} \\
 X_2 &= \chi^3 + i\chi^4 = 2\sqrt{2}\chi_{31}^\dagger = 2\sqrt{2}\chi_{24} \\
 X_3 &= \chi^5 + i\chi^6 = 2\sqrt{2}\chi_{12}^\dagger = 2\sqrt{2}\chi_{34}
 \end{aligned}
 \tag{3.8}$$

In terms of these degrees of freedom, the determinant takes the form

$$\det_4(e^{iB_{AB}}\chi_{AB}) = \begin{vmatrix} 0 & X_3^\dagger e^{-\frac{i\pi}{2}(\gamma_1+\gamma_2)} & -X_2^\dagger e^{\frac{i\pi}{2}(\gamma_3+\gamma_1)} & X_1 e^{\frac{i\pi}{2}(\gamma_2-\gamma_3)} \\ -X_3^\dagger e^{\frac{i\pi}{2}(\gamma_1+\gamma_2)} & 0 & X_1^\dagger e^{-\frac{i\pi}{2}(\gamma_2+\gamma_3)} & X_2 e^{\frac{i\pi}{2}(\gamma_3-\gamma_1)} \\ X_2^\dagger e^{-\frac{i\pi}{2}(\gamma_3+\gamma_1)} & -X_1^\dagger e^{\frac{i\pi}{2}(\gamma_2+\gamma_3)} & 0 & X_3 e^{\frac{i\pi}{2}(\gamma_1-\gamma_2)} \\ -X_1 e^{-\frac{i\pi}{2}(\gamma_2-\gamma_3)} & -X_2 e^{-\frac{i\pi}{2}(\gamma_3-\gamma_1)} & -X_3 e^{-\frac{i\pi}{2}(\gamma_1-\gamma_2)} & 0 \end{vmatrix}
 \tag{3.9}$$

It is evaluated to give

$$\begin{aligned}
 \det_4(e^{iB_{AB}}\chi_{AB}) &= \frac{1}{64} (|X_1|^2 + |X_2|^2 + |X_3|^2)^2 \\
 &\quad - \frac{1}{16} \sin^2(\pi\gamma_3) |X_1|^2 |X_2|^2 - \frac{1}{16} \sin^2(\pi\gamma_1) |X_2|^2 |X_3|^2 \\
 &\quad - \frac{1}{16} \sin^2(\pi\gamma_2) |X_3|^2 |X_1|^2
 \end{aligned}
 \tag{3.10}$$

We note that the expression above depends only on the three absolute values of $|X|$ and is independent of the three angles. We can further change variables as follows:

$$|X_i| = r \mu_i, \quad \sum_{i=1}^3 \mu_i^2 = 1
 \tag{3.11}$$

and write

$$\left(\frac{4\pi}{g} \frac{1}{\sqrt{2}}\right)^4 \det_4(e^{iB_{AB}}\chi_{AB}) = \left(\frac{\pi}{g}\right)^4 r^4 \left(1 - 4 \sin^2(\pi\gamma_3) \mu_1^2 \mu_2^2 - 4 \sin^2(\pi\gamma_1) \mu_2^2 \mu_3^2 - 4 \sin^2(\pi\gamma_2) \mu_3^2 \mu_1^2\right)
 \tag{3.12}$$

In summary after integrating out all of the ν and $\bar{\nu}$ fermionic collective coordinates we find the following generic instanton factor in the measure:

$$\begin{aligned}
 \mathcal{F}_{\text{inst}} &:= e^{-\frac{8\pi^2}{g^2} + i\theta} (1 - 4 \sin^2(\pi\gamma_3) \mu_1^2 \mu_2^2 - 4 \sin^2(\pi\gamma_1) \mu_2^2 \mu_3^2 - 4 \sin^2(\pi\gamma_2) \mu_3^2 \mu_1^2)^{N-2} \\
 &\equiv e^{2\pi i\tau_0} (1 - \mathcal{Q}(\mu_i, \gamma_i))^{N-2}
 \end{aligned}
 \tag{3.13}$$

This factor is integrated over the $AdS_5 \times S^5$ space spanned by x_0^μ , ρ and the five angles of χ^a

$$\int d^4 x_0 \frac{d\rho}{\rho^5} d^5 \hat{\chi} = (2\pi)^3 \int d^4 x_0 \frac{d\rho}{\rho^5} d\mu_1 d\mu_2 d\mu_3 \delta(\mu_1^2 + \mu_2^2 + \mu_3^2 - 1)
 \tag{3.14}$$

exactly as in [15, 18]. As we are interested in the limit $N \rightarrow \infty$ we can rewrite eq. (3.13) as a total exponent and evaluate the integrals over μ_i via a saddle-point approximation,

$$\int_{\mu_i} e^{2\pi i\tau_0} (1 - \mathcal{Q}(\mu_i))^{N-2} = \int_{\mu_i} \exp\left(2\pi i\tau_0 + (N-2) \log(1 - \mathcal{Q}(\mu_i))\right)$$

$$\approx \exp(2\pi i\tau_0 - N \mathcal{Q}(\mu_i|_{\text{saddle}})) \tag{3.15}$$

This method selects the dominant value of the function $\mathcal{Q}(\mu_i)$ to be $\mathcal{Q}(\mu_i|_{\text{saddle}}) \sim \frac{1}{N}$ and has therefore allowed us to expand the log to leading power in \mathcal{Q} in the last line.

What we have calculated so far is a large- N expression for the characteristic instanton factor

$$\mathcal{F}_{\text{inst}} = \exp(2\pi i\tau_0 - N \mathcal{Q}(\mu_i|_{\text{saddle}}, \gamma_i)) \tag{3.16}$$

This factor arises in an instanton calculation of a generic correlation function in gauge theory. When applied to the calculation of Yang-Mills correlators involving operators which are dual to the supergravity fields, the instanton result in gauge theory must match with the corresponding D-instanton contribution in string theory. This means that the characteristic factor (3.16) due to the Yang-Mills instanton must correspond to $\exp(2\pi i\tau)$, where τ is the dilaton-axion field in dual string theory.⁴ By matching exponents we read off the instanton prediction for the dilaton-axion field:

$$\tau = \tau_0 - \frac{N}{2\pi i} (4 \sin^2(\pi\gamma_3) \mu_1^2 \mu_2^2 + 4 \sin^2(\pi\gamma_1) \mu_2^2 \mu_3^2 + 4 \sin^2(\pi\gamma_2) \mu_3^2 \mu_1^2) \tag{3.17}$$

We note that this semi-classical field theory result is valid for any value of the parameters γ_i and, as such, can be interpreted [16] as a (weak-coupling) prediction for the τ field in the exact string theory background.

The supergravity regime is reached in the limit of $\gamma_i \ll 1$ which gives:

$$\tau \rightarrow \tau_0 + 2N\pi i (\gamma_3^2 \mu_1^2 \mu_2^2 + \gamma_1^2 \mu_2^2 \mu_3^2 + \gamma_2^2 \mu_3^2 \mu_1^2) \tag{3.18}$$

This precisely matches with the Frolov's three parameter supergravity solution (2.1) for the dilaton-axion field:

$$\tau = ie^{-\phi} + C \tag{3.19}$$

$$= ie^{-\phi_0} \left(1 + \hat{\gamma}_3^2 \mu_1^2 \mu_2^2 + \hat{\gamma}_1^2 \mu_2^2 \mu_3^2 + \hat{\gamma}_2^2 \mu_3^2 \mu_1^2\right)^{1/2} + C^0 \tag{3.20}$$

$$= \tau_0 + \frac{ie^{-\phi_0}}{2} (\hat{\gamma}_3^2 \mu_1^2 \mu_2^2 + \hat{\gamma}_1^2 \mu_2^2 \mu_3^2 + \hat{\gamma}_2^2 \mu_3^2 \mu_1^2)$$

where the deformation parameters are $\hat{\gamma}_i^2 = N g^2 \gamma_i^2$ and one identifies the coordinates on the deformed supergravity \tilde{S}^5 sphere with the χ -collective coordinates of the instanton.

It is clear in the above that an analogous calculation for the case of one *anti*-instanton would yield the same type of gauge/supergravity matching for the conjugate parameter $\bar{\tau}$. One can also extend this calculation to include the multi-instanton sectors, as in [15, 18]. In the large N limit the partition function in the k -instanton sector is:

$$\int d\mu_{\text{inst}}^k e^{-S_{\text{inst}}^k} = \frac{\sqrt{Ng^2}}{2^{33} \pi^{27/2}} \frac{k^{-7/2}}{g^2} \sum_{d|k} \frac{1}{d^2} \int \frac{d^4 x d\rho}{\rho^5} d^5 \hat{\Omega} \prod_{A=1,2,3,4} d^2 \xi^A d^2 \bar{\eta}^A e^{2\pi i k \tau} \tag{3.21}$$

where τ is given by the same eq. (3.17).

⁴More detail about instanton and D-instanton contributions to the string effective action can be found in [15, 18–20].

4. Complex β deformations

In this section we consider the more general case of marginal deformations with complex values of the deformation parameters $\beta_i \in \mathbb{C}$

$$\beta_1 = \gamma_1 + i\sigma_1, \quad \beta_2 = \gamma_2 + i\sigma_2, \quad \beta_3 = \gamma_3 + i\sigma_3 \quad (4.1)$$

The supergravity solution corresponding to this case was obtained in [7] by performing three consecutive $STsTS^{-1}$ transformations (where S is the S-duality) acting on the three natural tori of S^5 . This family of solutions is expected to be dual to a deformed Yang-Mills theory with three complex deformation parameters.

We will first explain how to extend the instanton calculation on the gauge theory side from real to complex β_i -deformations. We will carry out this calculation for arbitrary (not necessarily small) values of the deformation parameter $\beta_i \in \mathbb{C}$. The main result of this section is the instanton prediction for the dilaton-axion field τ . We will show that in the limit of small β_i it will match precisely with the τ field of Frolov's supergravity dual [7]. As before, the small- β_i limit is required to ensure the validity of the supergravity approximation to full string theory.

We now need to specify the deformed gauge theory. The absence of supersymmetry and the complex-valuedness of the deformations β_i make it difficult. It is not entirely clear how to uniquely define this theory and, more importantly, how to guarantee its marginality in the large N -limit.⁵ Fortunately, the instanton calculation which we are about to present does not require the full knowledge of the gauge-theory Lagrangian, beyond its gauge and Yukawa interactions specified below.

The instanton configuration at the leading order in weak coupling is defined as in equations (3.2)-(3.3) with the scalar field equation (3.3) taking the form:

$$\mathcal{D}^2\Phi^{AB} = \frac{h}{g}\sqrt{2}i(e^{i\pi B^{AB}}\lambda^A\lambda^B - e^{-i\pi B^{AB}}\lambda^B\lambda^A), \quad (4.2)$$

Here B_{AB} is a complex-valued matrix obtained from the one in (2.8) by the substitution $\gamma_i \rightarrow \beta_i$. The factor of h/g on the right hand side of (4.2) accounts for the change of the coupling constant from g to h in the Yukawa couplings, where h is a new complex parameter. We note that the resulting instanton configuration depends on h holomorphically.⁶

Following the approach of section 3 we integrate out fermionic collective coordinates ν_i^A and $\bar{\nu}_i^A$. For each value of $i = 1, \dots, N-2$ this integration gives a factor of the determinant (3.7) times an appropriate rescaling by h/g . We find

$$\left(\frac{1}{g}\right)^4 \det_4\left(e^{i\pi B^{AB}}\chi_{AB}\right) \longrightarrow \left(\frac{1}{g}\right)^4 \left(\frac{h}{g}\right)^2 \det_4\left(e^{i\pi B^{AB}}\chi_{AB}\right). \quad (4.3)$$

⁵The absence of supersymmetry prevents one from using the Leigh-Strassler approach [3] in terms of conformal constraints, while the complex-valuedness of the deformation parameters makes it difficult to use the star-product formulation.

⁶At leading order in g the dependence on h^* can come only through the equation conjugate to (4.2), which involves anti-fermion zero modes $\bar{\lambda}$ on the right hand side. These are vanishing in the instanton background. It is clear then that the anti-instanton configuration, will depend on h^* and not on h .

After evaluating this determinant, the result for the characteristic instanton factor in the large- N limit is:

$$\mathcal{F}_{\text{inst}} = \exp \left[2\pi i \tau_0 + 2N \log \left(\frac{h}{g} \right) + N \log (1 - \mathcal{Q}(\mu_i, \beta_i)) \right], \quad (4.4)$$

where $\mathcal{Q}(\mu_i, \beta_i)$ is the same function as before, but with the complex β_i parameters in place of real γ_i ,

$$\mathcal{Q}(\mu_i, \beta_i) = 4 \left(\sin^2(\pi\beta_3) \mu_1^2 \mu_2^2 + \sin^2(\pi\beta_1) \mu_2^2 \mu_3^2 + \sin^2(\pi\beta_2) \mu_3^2 \mu_1^2 \right) \quad (4.5)$$

By taking the small deformation limit, $|\beta_i|^2 \ll 1$, appropriate for comparison with the supergravity solution, we find

$$\mathcal{F}_{\text{inst}} = \exp \left[2\pi i \tau_r - 4\pi^2 N \left((\gamma_1^2 - \sigma_1^2 + 2i\gamma_1\sigma_1) \mu_2^2 \mu_3^2 + (\gamma_2^2 - \sigma_2^2 + 2i\gamma_2\sigma_2) \mu_1^2 \mu_3^2 + (\gamma_3^2 - \sigma_3^2 + 2i\gamma_3\sigma_3) \mu_2^2 \mu_1^2 \right) \right], \quad (4.6)$$

Here τ_r is the constant which has the meaning of the ‘renormalised’ Yang-Mills coupling as in section 8 of [15] and in [6]. It is defined via

$$\tau_r := \tau_0 - \frac{iN}{\pi} \log \frac{h}{g} \quad (4.7)$$

The dilaton and axion field components of the Frolov’s supergravity dual with three complex deformations are given by [7]:

$$e^\phi = e^{\phi_0} G^{1/2} H, \quad C = C^0 + e^{-\phi_0} H^{-1} Q, \quad (4.8)$$

where the expressions for the functions G , H and Q can be found in the Appendix B of [7]. By employing these expressions and (4.8) one can easily calculate the axion-dilaton field for the case of complex deformations. The result thus obtained reads

$$\begin{aligned} e^{2\pi i \tau} &= e^{2\pi i (ie^{-\phi} + C)} \\ &= \exp \left[-2\pi e^{-\phi_0} \left[1 + \frac{1}{2}(\hat{\gamma}_1^2 - \hat{\sigma}_1^2) \mu_2^2 \mu_3^2 + \frac{1}{2}(\hat{\gamma}_2^2 - \hat{\sigma}_2^2) \mu_1^2 \mu_3^2 + \frac{1}{2}(\hat{\gamma}_3^2 - \hat{\sigma}_3^2) \mu_2^2 \mu_1^2 \right] \right. \\ &\quad \left. + 2\pi i (C^0 + e^{-\phi_0} (\hat{\gamma}_1 \hat{\sigma}_1 \mu_2^2 \mu_3^2 + \hat{\gamma}_2 \hat{\sigma}_2 \mu_1^2 \mu_3^2 + \hat{\gamma}_3 \hat{\sigma}_3 \mu_2^2 \mu_1^2)) \right] \end{aligned} \quad (4.9)$$

By making the identification

$$\hat{\gamma}_i = g_r \sqrt{N} \gamma_i, \quad \hat{\sigma}_i = -g_r \sqrt{N} \sigma_i, \quad \tau_r = ie^{-\phi_0} + C^0 \quad (4.10)$$

one can immediately see this supergravity result is in perfect agreement with our field theory prediction (4.6).

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